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Reflection ordering on the group $G(m, m, n)$

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ABSTRACT

In the paper Shi (2008) [13], we introduced a partial ordering, called the reflection ordering, on the elements of $G(m, p, n)$ and described such an ordering on the groups $G(m, 1, n)$, $m \geqslant 1$. In the present paper, we describe the reflection ordering on the group $G(m, m, n)$. As a by-product, we obtain a formula for the enumeration of a certain subset in the symmetric group S_n, which is of independent combinatorial interest.

0. Introduction

Let $\mathbb{P}$ (respectively, $\mathbb{N}$, $\mathbb{Z}$) denote the set of positive integers (respectively, non-negative integers, integers). For any $k, n \in \mathbb{N}$ with $k \leqslant n$, let $[k, n] = \{k, k+1, \ldots, n\}$ and $[n] = [1, n]$.

0.1. The group G(m,p,n)

Fix $m, p, n \in \mathbb{P}$ with $p \mid m$, let $G(m, p, n)$ be the group consisting of all $n \times n$ monomial matrices w such that all the non-zero entries, say $\theta_1, \ldots, \theta_n$, of w are mth roots of unity in $\mathbb{C}$ with $\left(\prod_{i=1}^{n} \theta_i\right)^{m/p} = 1$. Any $w \in G(m, p, n)$ can be expressed in the form $w = [a_1, \ldots, a_n \mid \sigma]$ with $\sigma \in S_n$ and $a_k \in \mathbb{Z}$, where S_n is the symmetric group on the set $[n]$, and the entry in the $(k, (k)\sigma)$-position of w is $\exp\left(\frac{2\pi a_k \sqrt{-1}}{m}\right)$ for $k \in [n]$. Clearly, $p \mid \sum_{k=1}^{n} a_k$. Denote σ by $\pi(w)$. Then π is a surjective homomorphism from the group $G(m, p, n)$ to S_n.

0.2. Reflection groups

A *reflection* in a complex vector space V is by definition a linear transformation of V of finite order, whose fixed point subspace has codimension 1 in V. A reflection defined here is called a *pseudo-reflection* in some literature to distinguish it from the concept of a reflection in a Euclidean space. A group G generated by reflections is called a *reflection group*. A reflection group is called *irreducible* if it cannot be expressed as a direct product of two reflection subgroups. The irreducible finite reflection groups were classified in [11]. The group $G(m, p, n)$ is a reflection group under its natural action on the space $V = \mathbb{C}^n$, which has been studied extensively in the literature (see, for example, [7,11]).

0.3. Reflection ordering in G(m,p,n)

Any element w in a reflection group G can be expressed as a product of reflections in G. Define the *reflection length* $l_T(w)$ of w to be the smallest possible number of reflections occurring in such a product. For any $y, w \in G$, we say that w *covers* y (or y *is covered by* w), written $y \lessdot_T w$, if yw^{-1} is a reflection in G with $l_T(w) = l_T(y) + 1$. Let $\preceq_T$ be the partial ordering, called *reflection ordering*, on G, which is the transitive closure of the covering relation $\lessdot_T$. When $G = G(m, p, n)$, we denote l_T, $\lessdot_T$, $\preceq_T$ by l_{mp}, $\lessdot_{mp}$, $\preceq_{mp}$, respectively. In [13], we gave an explicit description for the ordering $\preceq_{m1}$ on $G(m, 1, n)$, $m \geqslant 1$.

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0.4. Our results on G(m,m,n)

The aim of the present paper is to describe the ordering $\preceq_{mm}$ on $G(m, m, n)$, $m > 1$, or equivalently, the set $B^{(m)}(w) = \{y \in G(m, m, n) \mid y \preceq_{mm} w\}$ for any $w \in G(m, m, n)$. Our results are Theorems 2.1, 2.3, 2.5–2.8, 2.10 and 4.4. More precisely, Theorems 2.1 and 2.3 are concerned with the set $B^{(m)}(w)$ in the case when the dimension of the fixed point space of $w \in G(m, m, n)$ in $\mathbb{C}^n$ is equal to the number of disjoint cycles in the decomposition of $\pi(w)$ (see ((1.2.1))): Theorem 2.1 gives the equation $B^{(m)}(w) = B^{(1)}(w)$ and Theorem 2.3 computes the cardinal of the set $B^{(m)}(w)$ for these elements w. Theorem 2.5 reduces the study of $B^{(m)}(w)$ to that of two simpler sets $B_1^{(m)}(w)$ and $B_2^{(m)}(w)$ (see 2.4). Then Theorems 2.6–2.7 are for the description of the sets $B_2^{(m)}(w)$ and $B_1^{(m)}(w)$ respectively. Theorem 2.8 gives a partition of the set $B^{(m)}(w)$, while Theorem 2.10 enumerates the set $B_2^{(m)}(w)$ and some of its subsets. Theorem 4.4 provides a closed formula for the cardinal of a certain subset of S_n, which is of independent combinatorial interest and is applied to enumerate the set $B_2^{(m)}(w)$ in Theorem 2.10.

On the other hand, some algorithms for finding elements of $B^{(m)}(w)$ are introduced in 3.11.

0.5. Some background and questions

The function l_T and the ordering $\preceq_T$ were first studied by Carter on elements of Weyl groups in [8]. Since then, the function l_T and the ordering $\preceq_T$ on elements of Coxeter groups have been studied by a number of people in algebra, geometry and combinatorics, etc.

Let V be the real space providing Tits representation of a Weyl group W with T its reflection set. For any $w \in W$, let $V^w = \{v \in V \mid (v)w = v\}$. Carter proved that $l_T(w) = \mathrm{codim}_V V^w$ and that $t \in T$ satisfies $t \preceq_T w$ if and only if $V^t \supseteq V^w$ (see [8, Lemma 2.8]). Carter's result and argument can be extended to the case where W is any finite Coxeter group.

Dyer proved that for any element w in a Coxeter group W, $l_T(w)$ is equal to the minimal number of simple reflections that must be deleted from a fixed reduced expression of w so that the resulting product is equal to the identity element e of W; Dyer also proved that $l_T(w)$ is the minimal length of a path in the (directed) Bruhat graph from the element e to w, and that $l_T(w)$ is determined by the polynomial $R_{e,w}$ of Kazhdan and Lusztig (see [9, Theorems 1.1–1.3]).

Let W_1 be a certain reflection subgroup of a finite Coxeter group W (including all parabolic subgroups). Howlett and Lehrer proved that the elements of minimal reflection length in any coset wW_1 are all conjugate, provided w normalises W_1 (see [10, Corollary4.2]).

Brady and Watt proved that for any finite Coxeter system (W, S) and any Coxeter element c of W, the interval $[e, c]$ under the ordering $\preceq_T$ is a lattice (see [5, Theorem 4.14], [6, Theorem 7.8]).

Let Φ be the root system of a finite Coxeter group W and $m \in \mathbb{N}$. By applying ordering $\preceq_T$ on W, Athanasiadis and Tzanaki proved that both the generalized cluster complex $\Delta^m(\Phi)$ and its positive part $\Delta_+^m(\Phi)$ are shellable and higher Cohen–Macaulay, verifying some conjectures by Fomin-Reading and Reiner (see [1, Theorem 1.1]).

By the virtue of ordering $\preceq_T$ on a finite Coxeter group W, Bessis defined the dual braid monoid $M(P_c)$ from (W, T, c) and proved that $M(P_c)$ is a Garside monoid, where c is a certain Coxeter element (see [2, Theorem 2.3.2]).

It is natural to ask the following question: to what extent the results for the function l_T and the ordering $\preceq_T$ on Coxeter groups could be generalized to the complex reflection groups? Bessis asked in [2, Subsection 6.4] if Coxeter groups admit a classification purely in terms of reflection length and if the ordering $\preceq_T$ has a geometric interpretation similar to the ones known for the Bruhat order. The same question could be asked to the complex reflection groups. Bessis (together with Corran) studied his question for the group $G(m, m, n)$ and answered it partially (see [3]). Then Bessis proved the lattice property of $[e, c]$ for all well generated complex reflection groups G, which include $G(m, m, n)$, where c is a certain Coxeter element of G (see [4, Proposition 8.8 and Lemma 8.9]).

0.6. Contents

The contents are organized as follows. Section 1 contains preliminaries: some concepts and results are collected for later use. Subsequent sections are for the description of the set $B^{(m)}(w)$ for any $w \in G(m, m, n)$. We state our main results on the set $B^{(m)}(w)$ in Section 2 and give the proofs and some algorithms in Section 3. Finally we enumerate a certain subset of $B_2^{(m)}(w)$ in Section 4.

1. Preliminaries

We collect some concepts and results in this section for later use.

1.1. Reflections in G(m,p,n)

The group $G(m, p, n)$ naturally acts on the space $V = \mathbb{C}^n$. An element $w = [a_1, \ldots, a_n \mid \sigma] \in G(m, p, n)$ is a reflection if and only if one of the following conditions holds:

(1) $\sigma = (i, j)$ is a transposition of i and j for some $i < j$ in $[n]$, $a_j \equiv -a_i$ and $a_k \equiv 0 \pmod m$ for $k \neq i, j$. Denote w by $t(i, j; a_i)$ or $t(j, i; -a_i)$.

(2) $w \neq 1$ is diagonal with $n-1$ diagonal entries being 1 (such kind of reflections exist only when $p < m$). Denote w by $s(k; a_k)$ if $a_k \neq 0$.

The reflections of $G(m, m, n)$ are precisely those satisfying condition (1); while all the reflections of $G(m, p, n)$ not in $G(m, m, n)$ satisfy condition (2).

1.2. $t_m(w)$ and $t_0(w)$

By *a partition* $E = \{E_1, \ldots, E_l\}$ of a set X, we mean a disjoint union $X = \bigcup_{i\in[l]} E_i$ of nonempty subsets $E_1, \ldots, E_l$, call E_i a *part* of E.

Let $C = (c_1, c_2, \ldots, c_r) \in \mathbb{P}^r$. A subset E of $[r]$ is called (C, m)-*perfect* if $\sum_{h\in E} c_h \equiv 0 \pmod m$. A partition $P = \{P_1, \ldots, P_l\}$ of $[r]$ is called (C, m)-*admissible* if all the P_j's are (C, m)-perfect. Let $\Lambda(C; m)$ be the set of all (C, m)-admissible partitions of $[r]$. Then $\Lambda(C; m) \neq \emptyset$ if and only if $\sum_{h\in[r]} c_h \equiv 0 \pmod m$. Define $t_m(P) = |P|$ for any $P \in \Lambda(C; m)$ and $t(C, m) = \max\{t_m(P) \mid P \in \Lambda(C; m)\}$ if $\Lambda(C; m) \neq \emptyset$.

For any $w = [a_1, \ldots, a_n \mid \sigma] \in G(m, 1, n)$, write σ as a product of disjoint cyclic permutations:

$$\sigma = \sigma_1\sigma_2\cdots\sigma_r, \tag{1.2.1}$$

where $\sigma_k = (i_{k1}, i_{k2}, \ldots, i_{km_k})$ for $k \in [r]$ with $\sum_{j\in[r]} m_j = n$. Set $r(w) = r$ and $|\sigma_k| = m_k$. Let $I_j(w) = \{i_{j1}, i_{j2}, \ldots, i_{jm_j}\}$ for $j \in [r]$. Then $I(w) = \{I_1(w), \ldots, I_r(w)\}$ is a partition of $[n]$ determined by w. Let $c_j = \sum_{k\in I_j(w)} a_k$ and let $C(w) = (c_1, c_2, \ldots, c_r)$.

For any $w \in G(m, 1, n)$, let $t_0(w) = {}^{\#}\{j \in [r] \mid c_j \equiv 0 \pmod m\}$ and $V^w = \{v \in V \mid (v)w = v\}$. Then $t_0(w) = \dim_V V^w$.

For $w \in G(m, m, n)$, we always have $\Lambda(C(w); m) \neq \emptyset$ since the partition $\{[r]\}$ lies in $\Lambda(C(w); m)$. Denote $t_m(w) = t(C(w), m)$ if $w \in G(m, m, n)$.

Whenever it is applicable, the numbers $t_m(w)$ and $t_0(w)$ depend only on w, but not on the choice of an r-tuple $C(w) = (c_1, c_2, \ldots, c_r)$, hence they are well-defined.

1.3. Standard decomposition of $w \in G(m, 1, n)$

Keep the notation in 1.2 for $w = [a_1, \ldots, a_n \mid \sigma] \in G(m, 1, n)$, in particular, the expression (1.2.1) for σ. For any $k \in [r]$, define $w_k = [a_{k1}, \ldots, a_{kn} \mid \sigma_k] \in G(m, 1, n)$ by setting

$$a_{kj} = \begin{cases} a_j, & \text{if } j \in I_k, \\ 0, & \text{if otherwise.} \end{cases}$$

Then $w_jw_k = w_kw_j$ for any $j, k \in [r]$ and

$$w = w_1w_2\cdots w_r. \tag{1.3.1}$$

Call (1.3.1) *a standard decomposition of w into cyclic components*.

We say that w_i is *perfect* if $\sum_{j=1}^n a_{ij} \equiv 0 \pmod m$ (i.e., $w_i \in G(m, m, n)$) and that w_i, w_j with $i \neq j$ are *perfect-in-pair*, if neither w_i nor w_j is perfect, and $\sum_{h=1}^n (a_{ih} + a_{jh}) \equiv 0 \pmod m$. We have $t_0(w) = {}^{\#}\{j \in [r] \mid w_j \text{ perfect}\}$.

Lemma 1.4 (*see [13, Lemma 3.7]*). *(1) $y, w \in G(m, 1, n)$ satisfy $y \preceq_{m1} w$ if and only if there exist some standard decompositions $w = w_1w_2\cdots w_r$, $y = y_1y_2\cdots y_t$ into cyclic components such that the following conditions hold:*

(a) There are some $u \in [0, r]$ with $r - u$ even such that all the $\{w_j, w_{j+1}\}$ are perfect-in-pair for $u < j < r$ with $j - u$ odd and that no two w_h, w_k with $h \neq k$ in $[u]$ are perfect-in-pair.

(b) There are some $1 \leqslant p_1 < p_2 < \cdots < p_{(r+u)/2} = t$ such that $y_{p_{i-1}+1}y_{p_{i-1}+2}\ldots y_{p_i} \preceq_{m1} w_i$ for $i \in [u]$ and $y_{p_{j-1}+1}y_{p_{j-1}+2}\ldots y_{p_j} \preceq_{m1} w_{2j-u+1}w_{2j-u+2}$ for $u \leqslant j < (r+u)/2$, where we stipulate $p_0 = 0$.

(2) If $y, w \in G(m, 1, n)$ satisfy $y \preceq_{m1} w$, then $r(y) - t_0(y) \leqslant r(w) - t_0(w)$.

1.5. The set $B^{(p)}(w)$

For any $w \in G(m, p, n)$, define $B^{(p)}(w) = \{y \in G(m, p, n) \mid y \preceq_{mp} w\}$. The following is a consequence of [13, Theorem 3.9(1)] (Note that $G(m, m, n) \subseteq G(m, 1, n)$).

Theorem 1.6. *Let $w = [a_1, \ldots, a_n \mid \sigma] \in G(m, m, n)$ be with $\sigma = (1, 2, \ldots, r)$ a cyclic permutation and $a_j = 0$ for $j > r$. Then $\left|B^{(1)}(w)\right| = \frac{1}{r+1}\binom{2r}{r}$.*

The following result is concerned with the formula for the reflection lengths of an element in $G(m, 1, n)$ and in $G(m, m, n)$.

Theorem 1.7 (*see [12, Theorems 3.1 and 2.1]*). *We have $l_{mm}(w) = n + r(w) - 2t_m(w)$ for $w \in G(m, m, n)$ and $l_{m1}(w) = n - t_0(w)$ for $w \in G(m, 1, n)$.*

1.8. Covering relation in G(m,1,n)

By Theorem 1.7, we see that $y, w \in G(m,1,n)$ satisfy the covering relation $y \lessdot_{m1} w$ (see 0.3) if and only if there exist some standard decompositions of $w = w_1 w_2 \cdots w_r$ and $y = y_1 y_2 \cdots y_t$ into cyclic components such that one of the following conditions holds:

(i) $t = r+1$ and $w_i = y_i$ for all $i \in [r-1]$, and $w_r^{-1} y_r y_{r+1}$ is a reflection of the form $t(h,k;a)$ for some $h \neq k$ in $[n]$ and $a \in \mathbb{Z}$, where the number of perfect elements in y_r, y_{r+1} is greater than that in w_r, i.e., either both y_r, y_{r+1} are perfect, or w_r is not perfect but one of y_r, y_{r+1} is perfect.
(ii) $t = r$ and $w_i = y_i$ for all $i \in [r-1]$, and $w_r^{-1} y_r$ is a reflection of the form $s(h;a)$ for some $h \in [n]$ and $a \in \mathbb{Z}$ with $a \not\equiv 0 \pmod{m}$, where y_r is perfect but w_r is not.
(iii) $t = r-1$ and $w_i = y_i$ for $i \in [r-2]$, and $y_{r-1}^{-1} w_{r-1} w_r$ is a reflection of the form $t(h,k;a)$ for some $h \neq k$ in $[n]$ and $a \in \mathbb{Z}$ with $a \not\equiv 0 \pmod{m}$, where y_{r-1} is perfect but neither w_{r-1} nor w_r is perfect.

The following result is a consequence of [12, Proposition 5.3].

Proposition 1.9. *For any $w \in G(m,m,n)$ with $r = r(w)$, let $C(w) = (c_1, \ldots, c_r)$ be an r-tuple obtained from $I(w)$ as in 1.2. Let $\Lambda_0(C(w);m) = \{P \in \Lambda(C(w);m) \mid t_m(P) = t_m(w)\}$. We have $l_{mp}(w) = \mathrm{codim}_V V^w$ if and only if there exists some $P = \{P_1, P_2, \ldots, P_s\} \in \Lambda_0(C(w);m)$ with $|P_i| \leqslant 2$ for any $i \in [s]$ such that $|P_i| = 2$ only if P_i is $(C(w),m)$-perfect (see 1.2).*

2. Our results on the set $B^{(m)}(w)$

In the subsequent sections, we focus our attention to the ordering $\preceq_{mm}$ on $G(m,m,n)$.

In this section, we state our results on the set $B^{(m)}(w)$ for $w \in G(m,m,n)$ (see 0.4). Theorems 2.1, 2.5, 2.8 and 2.10 will be proved in Section 3. The proofs for all the other results can be deduced directly either from the definitions or by some previously stated results, and hence are left to the readers.

There is no simple relation between the sets $B^{(1)}(w)$ and $B^{(m)}(w)$ in general. However, we have the following

Theorem 2.1. *Let $w \in G(m,m,n)$. Then $B^{(1)}(w) = B^{(m)}(w)$ if and only if $r(w) = t_0(w)$. When the equivalent conditions hold, we have $l_{mm}(w) = \mathrm{codim}_V V^w$ (see 1.2).*

Remark 2.2. For $w \in G(m,m,n)$ with $r(w) = t_0(w)$, the set $B^{(m)}(w) = B^{(1)}(w)$ can be described in a way similar to that in [13, Subsection 3.18–3.19] by Lemma 1.4.

Recall the map $\pi : G(m,p,n) \to S_n$ defined in 0.1. By Theorem 2.1, we see that for $w \in G(m,m,n)$ with $\pi(w)$ a cyclic permutation, the assertion of Theorem 1.6 holds with $B^{(m)}(w)$ in the place of $B^{(1)}(w)$. By Lemma 1.4, we further have

Theorem 2.3. *For $w \in G(m,m,n)$ with $r = r(w) = t_0(w)$, write $\pi(w) = \sigma_1\sigma_2\cdots\sigma_r$ as in (1.2.1) with $|\sigma_k| = m_k$. Then*

$$\left|B^{(m)}(w)\right| = \prod_{k=1}^{r} \frac{1}{m_k+1}\binom{2m_k}{m_k}. \tag{2.3.1}$$

2.4. The sets $B_1^{(m)}(w)$ and $B_2^{(m)}(w)$

By Theorem 1.7, we see that $y, w \in G(m,m,n)$ satisfy the covering relation $y \lessdot_{mm} w$ (see 0.3) if and only if the following two conditions are satisfied:

(a) yw^{-1} is a reflection, and
(b) either (i) $r(w) + 1 = r(y)$ and $t_m(w) + 1 = t_m(y)$, or
(ii) $r(w) = r(y) + 1$ and $t_m(w) = t_m(y)$.

For any $y, w \in G(m,m,n)$ with $y \lessdot_{mm} w$, we write $y \lessdot_1 w$ (respectively, $y \lessdot_2 w$) if w, y are in the case (b)(i) (respectively, (b)(ii)). For any $w \in G(m,m,n)$, let $B_1^{(m)}(w)$ (respectively, $B_2^{(m)}(w)$) be the set of all elements $y \in G(m,m,n)$ satisfying: there exists a sequence of elements $y_0 = y, y_1, \ldots, y_t = w$ with some $t \geqslant 0$ such that $y_{i-1} \lessdot_1 y_i$ (respectively, $y_{i-1} \lessdot_2 y_i$) for every $i \in [t]$. Then we have

Theorem 2.5.

$$B^{(m)}(w) = \bigcup_{x \in B_2^{(m)}(w)} B_1^{(m)}(x) \tag{2.5.1}$$

for any $w \in G(m,m,n)$.

By Theorem 2.5, for any $w \in G(m,m,n)$, to describe the set $B^{(m)}(w)$, we need only to describe the sets $B_1^{(m)}(w)$ and $B_2^{(m)}(w)$ separately.

The description of the set $B^{(m)}(w)$ can be reduced to the case of $t_m(w) = 1$ (see Lemma 3.3). The following result can be proved directly from the definition of the set $B_2^{(m)}(w)$.

Theorem 2.6. *Let $w \in G(m, m, n)$ be with $t_m(w) = 1$. Then an element $y \in G(m, m, n)$ is in the set $B_2^{(m)}(w)$ if and only if there exists a sequence $x_0 = w, x_1, \ldots, x_u = y$ in $G(m, m, n)$ with $u = l_{mm}(w) - l_{mm}(y)$ such that $\pi(x_{i-1}) \lessdot_{11} \pi(x_i)$, and $x_{i-1}^{-1}x_i$ is a reflection for every $i \in [u]$. When the equivalent conditions hold, we have $l_{mm}(w) - l_{mm}(y) = r(w) - r(y)$ and $t_m(y) = 1$.*

The next result, which follows directly from the definition of the set $B_1^{(m)}(w)$ and Theorem 1.7, is concerned with the set $B_1^{(m)}(w)$ for any $w \in G(m, m, n)$.

Theorem 2.7. *For any $w \in G(m, m, n)$, an element $y \in G(m, m, n)$ is in $B_1^{(m)}(w)$ if and only if there exists a sequence of elements $x_0 = y, x_1, \ldots, x_t = w$ in $G(m, m, n)$ with $t = l_{mm}(w) - l_{mm}(y)$ such that $\pi(x_{i-1}) \lessdot_{11} \pi(x_i)$ and $x_{i-1}^{-1}x_i$ is a reflection for every $i \in [t]$. When the equivalent conditions hold, we have $l_{mm}(w) - l_{mm}(y) = r(y) - r(w) = t_m(y) - t_m(w)$.*

For any $y, w \in G(m, m, n)$ with $y \preceq_{mm} w$, there exists some $z \in B_2^{(m)}(w)$ with $y \in B_1^{(m)}(z)$ by Theorem 2.5. By showing the inclusion $r(z) \in [t_m(w), r(w)]$, we get a partition of the set $B^{(m)}(w)$ as follows.

Theorem 2.8.

$$B^{(m)}(w) = \coprod_{k \in [t_m(w), r(w)]} B^{(m)}(w)_k, \quad \text{where } B^{(m)}(w)_k = \bigcup_{\substack{x \in B_2^{(m)}(w) \\ r(x) = k}} B_1^{(m)}(x).$$

Remark 2.9. For $w \in G(m, m, n)$, the condition $r(w) = t_m(w)$ is equivalent to $r(w) = t_0(w)$. When the equivalent conditions hold, we have $B_1^{(m)}(w) = B^{(m)}(w) = B^{(1)}(w)$ by Theorems 2.8 and 2.1.

For any $\sigma \in S_n$ and $k \in [r(\sigma)]$, let $U(\sigma) = \{\tau \in S_n \mid \sigma \preceq_{11} \tau\}$ and $U(\sigma)_k = \{\tau \in U(\sigma) \mid r(\tau) = k\}$. Then we have

Theorem 2.10. *Assume that $w \in G(m, m, n)$ satisfies $t_m(w) = 1$.*

(1) *$|B_2^{(m)}(w) \cap \pi^{-1}(\tau)| = m^k$ for any $\tau \in S_n$ with $\pi(w) \preceq_{11} \tau$ and $k = r(w) - r(\tau)$.*
(2) *$|B_2^{(m)}(w)| = \sum_{k \in [r]} m^{r-k} \cdot |U(\pi(w))_k|$ with $r = r(w)$.*

We shall deduce a formula of the cardinal $|U(\sigma)_k|$ of $U(\sigma)_k$ for any $\sigma \in S_n$ and any $k \in [r(\sigma)]$ in Theorem 4.4.

3. Proofs and algorithms

In the present section, we prove Theorems 2.1, 2.5, 2.8 and 2.10. Also, we introduce some algorithms for finding the elements in $B_i^{(m)}(w)$, $i = 1, 2$, for any $w \in G(m, m, n)$ (see 3.11).

3.1. A standard decomposition of $w \in G(m, m, n)$

Keep the notation in 1.2 for $w = [a_1, \ldots, a_n \mid \sigma] \in G(m, m, n)$, in particular, the expression (1.2.1) for σ and an $r(w)$-tuple $C(w)$. Let $P(w) = \{P_1(w), \ldots, P_t(w)\}$ be a partition of $[r(w)]$ in $\Lambda_0(C(w); m)$ (see Proposition 1.9) with $t = t_m(w)$. For any $k \in [t]$, define $w_k = [a_{k1}, \ldots, a_{kn} \mid \tau_k] \in G(m, m, n)$ by setting

$$a_{kj} = \begin{cases} a_j, & \text{if } j \in I_h \text{ for some } h \in P_k(w), \\ 0, & \text{if otherwise.} \end{cases} \qquad \tau_k = \prod_{h \in P_k(w)} \sigma_h. \tag{3.1.1}$$

Then $w_j w_k = w_k w_j$ for any $j, k \in [t]$ and

$$w = w_1 w_2 \cdots w_t. \tag{3.1.2}$$

Each w_k is called *a perfect factor* of w. Call (3.1.2) *a standard decomposition of w into perfect factors of maximal number.*

3.2. Covering relation in $G(m, m, n)$

By 2.4 and Theorem 1.7, we see that if $y, w \in G(m, m, n)$ satisfy the covering relation $y \lessdot_{mm} w$ (see 0.3) then there exist some standard decompositions of $w = w_1 w_2 \cdots w_t$ and $y = y_1 y_2 \cdots y_u$ into perfect factors of maximal number (hence $t = t_m(w)$ and $u = t_m(y)$) such that

(i) when $y \lessdot_1 w$ (see 2.4), we have $u = t + 1$ and $w_i = y_i$ for any $i \in [t-1]$ and that $w_t^{-1} y_t y_{t+1} = t(h, k; c)$, where $I_\alpha(w) = I_\beta(y) \cup I_\gamma(y)$ and $h \in I_\beta(y)$ and $k \in I_\gamma(y)$ and $\alpha \in P_t(w)$ and $\beta \in P_t(y)$ and $\gamma \in P_{t+1}(y)$.
(ii) when $y \lessdot_2 w$ (see 2.4), we have $u = t$ and $w_i = y_i$ for any $i \in [t-1]$ and that $w_t^{-1} y_t = t(h, k; c)$, where $I_\beta(w) \cup I_\gamma(w) = I_\alpha(y)$ and $\beta \neq \gamma$ in $P_t(w)$ and $\alpha \in P_t(y)$ and $h \in I_\beta(w)$ and $k \in I_\gamma(w)$.

By repeatedly applying (i)–(ii), we get

Lemma 3.3. *For $w \in G(m, m, n)$, we have $y \in B^{(m)}(w)$ if and only if there exist some $r(w)$-tuple $C(w)$ (respectively, $r(y)$-tuple $C(y)$) obtained from $I(w)$ (respectively, $I(y)$) as that in 1.2, and some $\{P_1, \ldots, P_t\} \in \Lambda_0(C(w); m)$, $\{Q_1, \ldots, Q_u\} \in \Lambda_0(C(y); m)$ with $t = t_m(w)$ and $u = t_m(y)$ such that $P_i(w) = \bigcup_{\beta \in K_i} Q_\beta(y)$ for any $i \in [t]$, where $P_h(w) = \bigcup_{j \in P_h} I_j(w)$ and $Q_k(y) = \bigcup_{j \in Q_k} I_j(y)$, and $[u] = \bigcup_{i \in [t]} K_i$ is a partition of the set $[u]$ with each K_i non-empty.*

By Lemma 3.3, we see that Theorem 2.6 can be used to describe the set $B^{(m)}(w)$ for any $w \in G(m, m, n)$ (i.e., not only the case of $t_m(w) = 1$, see 3.11).

3.4. *Proof of Theorem 2.1*

The last assertion follows by Proposition 1.9. Now we show the first assertion.

Assume that $w \in G(m, m, n)$ satisfies $r(w) = t_0(w)$. Then any $y \in G(m, m, n)$ with $y \lessdot_{mm} w$ must be in the case of 3.2(i) (or equivalently, never in the case of 3.2(ii)) with $P_t(w) = \{\alpha\}$ and $P_t(y) = \{\beta\}$ and $P_{t+1}(y) = \{\gamma\}$. Hence $r(y) = r(w) + 1 = t_0(w) + 1 = t_0(y)$. This in turn implies that all the elements $y \in B^{(m)}(w)$ satisfy $r(y) = t_0(y)$. We have $t_m(y) = t_0(y)$ and hence $l_{m1}(y) = l_{mm}(y)$ for any $y \in B^{(m)}(w)$ by Theorem 1.7. So the inclusion $B^{(m)}(w) \subseteq B^{(1)}(w)$ holds in general. Lemma 1.4(2), together with the assumption $r(w) = t_0(w)$, implies that any $y \in G(m, 1, n)$ with $y \lessdot_{m1} w$ satisfies the equation $r(y) = t_0(y)$, hence y, w must be in one of the cases (i), (iii) of 1.8 (or equivalently, never in the case (ii) of 1.8), and so $w^{-1}y$ must be a reflection in $G(m, m, n)$. This implies the equation $B^{(1)}(w) = B^{(m)}(w)$. On the other hand, assume that $w = [a_1, \ldots, a_n \mid \sigma] \in G(m, m, n)$ satisfies $r(w) \neq t_0(w)$. Then regarding w as an element in $G(m, 1, n)$, there exists a standard decomposition $w = w_1 w_2 \cdots w_r$ into cyclic components with w_1 not perfect (see 1.3). For any $i \in I_1(w)$, $w_1 s(i; c)$ is perfect for some $c \in [m-1]$ (here we use the notation in 1.2–1.3). Then $ws(i; c) \in B^{(1)}(w) \setminus B^{(m)}(w)$. So $B^{(1)}(w) \neq B^{(m)}(w)$. Our proof is complete. □

3.5. *Proof of Theorem 2.5*

We must show that for any $y \in B^{(m)}(w)$, there exists some $x \in B_2^{(m)}(w)$ with $y \in B_1^{(m)}(x)$. Clearly, it suffices to prove that if $w, x, y \in G(m, m, n)$ satisfy $x \lessdot_2 y \lessdot_1 w$ then there always exists some $y' \in G(m, m, n)$ with $x \lessdot_1 y' \lessdot_2 w$. Now assume $x \lessdot_2 y \lessdot_1 w$ in $G(m, m, n)$. Then there exist some reflections, say $t(i, j; a)$ and $t(h, k; b)$ with $y = w \cdot t(i, j; a)$ and $x = y \cdot t(h, k; b)$. There are some $\alpha, \beta, \gamma \in [r(w)]$ with $\beta \neq \gamma$ such that $i, j \in I_\alpha(w)$ and $h \in I_\beta(w)$ and $k \in I_\gamma(w)$. Clearly, $\{i, j\} \neq \{h, k\}$. If $\{i, j\} \cap \{h, k\} = \emptyset$, then $t(i, j; a)$ and $t(h, k; b)$ commute. Take $y' = w \cdot t(h, k; b)$. Then $x = y' \cdot t(i, j; a)$ and $x \lessdot_1 y' \lessdot_2 w$. If $\{i, j\} \cap \{h, k\}$ contains exactly one element, let us assume $j = h$ (the other cases can be dealt with similarly). Then $\alpha = \beta \neq \gamma$. Take $y' = w \cdot t(i, k; a + b)$. We have $x = y' \cdot t(i, j; a)$ and $x \lessdot_1 y' \lessdot_2 w$. The result is proved. □

Example 3.6. In the group $G(12, 12, 6)$, take

$$w = [2, 3, 2, 9, 9, 11 \mid (1456)(23)], \qquad y = [2, 3, 2, 10, 9, 10 \mid (14)(23)(56)],$$
$$y' = [2, 3, 3, 9, 9, 10 \mid (145623)], \qquad x = [2, 3, 2, 10, 9, 10 \mid (14)(2356)].$$

Then $y = w \cdot t(1, 5; 1)$ and $y' = w \cdot t(1, 2; 1)$ and

$$x = w \cdot t(1, 5; 1)\, t(5, 2; 0) = w \cdot t(1, 2; 1)\, t(1, 5; 1)$$

with $t_m(w) = t_m(y') = 1$ and $t_m(y) = t_m(x) = 2$. Hence

$$x \lessdot_2 y \lessdot_1 w \quad \text{and} \quad x \lessdot_1 y' \lessdot_2 w.$$

3.7. *Proof of Theorem 2.8*

For any $y, w \in G(m, m, n)$ with $y \preceq_{mm} w$, there exists some $z \in B_2^{(m)}(w)$ with $y \in B_1^{(m)}(z)$ by Theorem 2.5. Then

$$2l_{mm}(\pi(z)) - l_{mm}(\pi(w)) - l_{mm}(\pi(y)) = l_{mm}(w) - l_{mm}(y)$$

by Theorems 2.6 and 2.7. Hence by Theorem 1.7, we have

$$\begin{aligned} r(w) &\geqslant r(z) \\ &= n - l_{mm}(\pi(z)) \\ &= n - \frac{1}{2}(l_{mm}(\pi(w)) + l_{mm}(\pi(y)) + l_{mm}(w) - l_{mm}(y)) \\ &= n - \frac{1}{2}(n - r(w) + n - r(y) + n + r(w) - 2t_m(w) - n - r(y) + 2t_m(y)) \\ &= t_m(w) + r(y) - t_m(y). \end{aligned}$$

The value $r(z)$ is determined entirely by w, y. By the fact $r(y) - t_m(y) \geqslant 0$, we have $r(z) \in [t_m(w), r(w)]$. This proves our result. □

3.8. Proof of Theorem 2.10

For (1), we first observe that if $y \in G(m, m, n)$ satisfy $y \lessdot_2 w$ and $yw^{-1} = t(i, j; a)$, then $t_m(y) = 1$ by Theorem 2.6. So $z \in G(m, m, n)$ satisfies $z \lessdot_2 w$ and $\pi(z) = \pi(y)$ if and only if $wz^{-1} = t(i, j; b)$ with some $b \in [m]$. In general, suppose that $x_0 = y, x_1, \ldots, x_r = w$ in $G(m, m, n)$ satisfy $x_{i-1} \lessdot_2 x_i$ for every $i \in [r]$. Then $x_{i-1}x_i^{-1} = t(h_i, k_i; a_i)$ is a reflection with $\pi(x_i) \lessdot_{11} \pi(x_{i-1})$ for every $i \in [r]$. Any $z \in G(m, m, n)$ lies in $B_2^{(m)}(w) \cap \pi^{-1}(\pi(y))$ if and only if $z = t(h_r, k_r; b_r)\, t(h_{r-1}, k_{r-1}; b_{r-1}) \cdots t(h_1, k_1; b_1) \cdot w$ for some $b_1, \ldots, b_r \in [m]$. Let $\mathbf{b} = (b_1, \ldots, b_r)$ and $\alpha(\mathbf{b}) = t(h_r, k_r; b_r)\, t(h_{r-1}, k_{r-1}; b_{r-1}) \cdots t(h_1, k_1; b_1)$. Since the reflection lengths of both $\alpha(\mathbf{b})$ (in $G(m, m, n)$) and $\pi(\alpha(\mathbf{b}))$ (in S_n) are r, we see by [13, Lemma 3.15(5) and Subsection 3.18(1)] that for any $\mathbf{b}, \mathbf{c} \in [m]^r$, the equation $\alpha(\mathbf{b}) = \alpha(\mathbf{c})$ holds if and only if $\mathbf{b} = \mathbf{c}$. This proves (1). Under the assumption of $t_m(w) = 1$, we have $\pi^{-1}(\tau) \cap B_2^{(m)}(w) \neq \emptyset$ for any $\tau \in U(\pi(w))$. So (2) follows from (1) and Theorem 2.6 immediately. $\square$

In the remaining part of the section, we shall formulate some algorithms for finding the elements in the set $B^{(m)}(w)$ ($w \in G(m, m, n)$) and for checking the relation $y \preceq_{mm} w$ on $G(m, m, n)$. In order to do this, let us first introduce some concepts.

3.9. Reflection distance

For any $y, w \in G(m, m, n)$, define

$$\mathrm{dist}_m(y, w) = l_{mm}(yw^{-1}),$$

and call it the *reflection distance* between y and w. Clearly,

$$\begin{aligned} \mathrm{dist}_m(y, w) &= \mathrm{dist}_m(w, y) = \mathrm{dist}_m(y^{-1}, w^{-1}) \quad \text{and} \\ \mathrm{dist}_m(y, w) &\geqslant \mathrm{dist}_1(\pi(y), \pi(w)) \end{aligned}$$

hold in general.

The following result on reflection distance is important to our algorithms.

Lemma 3.10. *Let $w = [a_1, \ldots, a_n \mid \sigma]$ and $y = [c_1, \ldots, c_n \mid \tau]$ be in $G(m, m, n)$. Write*

$$\tau\sigma^{-1} = \eta_v \cdots \eta_2\eta_1, \quad \text{where } \eta_i = (h_{i1}, h_{i2}, \ldots, h_{in_i}), \forall i \in [v], \tag{3.10.1}$$

as a product of disjoint cyclic permutations with $\sum_{i=1}^{v} n_i = n$. Set $I_j = \{h_{j1}, h_{j2}, \ldots, h_{jn_j}\}$ for $j \in [v]$.

(1) $\mathrm{dist}_1(\sigma, \tau) = n - v$.
(2) $y \preceq_{mm} w$ *if and only if* $\mathrm{dist}_m(y, w) = l_{mm}(w) - l_{mm}(y)$.
(3) $\mathrm{dist}_m(y, w) = \mathrm{dist}_1(\tau, \sigma)$ *if and only if*

$$\sum_{k \in I_j} a_k \equiv \sum_{k \in I_j} c_k \pmod{m}, \quad \forall j \in [v]. \tag{3.10.2}$$

Proof. (1) follows by Theorem 1.7.

(2) The implication "$\Longrightarrow$" is obvious. Now we prove the implication "$\Longleftarrow$". Let $u = \mathrm{dist}_m(y, w)$. Then there exists a sequence of elements $x_0 = y, x_1, \ldots, x_u = w$ in $G(m, m, n)$ such that $x_{i-1}x_i^{-1}$ is a reflection for every $i \in [u]$. Since

$$u = l_{mm}(w) - l_{mm}(y) = \sum_{i=1}^{u}(l_{mm}(x_i) - l_{mm}(x_{i-1})) \leqslant \sum_{i=1}^{u} 1 = u,$$

we must have $l_{mm}(x_i) - l_{mm}(x_{i-1}) = 1$ for all $i \in [u]$. This implies $x_{i-1} \preceq_{mm} x_i$ for all $i \in [u]$ and hence $y \preceq_{mm} w$.

(3) Write $\tau\sigma^{-1}$ as in (3.10.1). Regarded as an element of $G(m, 1, n)$, yw^{-1} has a standard decomposition into cyclic components: $yw^{-1} = w_v \cdots w_2w_1$ with $\pi(w_i) = \eta_i$, $i \in [v]$ (see 1.3). For any $j \in [v]$, we have

$$\sum_{k \in I_j} a_k \equiv \sum_{k \in I_j} c_k \pmod{m} \Longleftrightarrow w_j \in G(m, m, n). \tag{3.10.3}$$

So Condition (3.10.2) implies that

$$\mathrm{dist}_m(y, w) = l_{mm}(yw^{-1}) = \sum_{i=1}^{v} l_{mm}(w_i) = \sum_{i=1}^{v} l_{11}(\eta_i) = l_{11}(\tau\sigma^{-1}) = \mathrm{dist}_1(\tau, \sigma)$$

by Theorem 1.7 and the fact that $\pi(w_j) = \eta_j$ is a cyclic permutation for any $j \in [v]$. Conversely, suppose that $\mathrm{dist}_m(y, w) = \mathrm{dist}_1(\tau, \sigma)$. Then

$$n - v = l_{11}(\tau\sigma^{-1}) = l_{mm}(yw^{-1}) = n + v - 2t_m(yw^{-1}) \geqslant n + v - 2v = n - v$$

by Theorem 1.7, result (1) and the fact that $t_m(yw^{-1}) \leqslant r(yw^{-1}) = v$. This implies the equation $t_m(yw^{-1}) = v$ and further the equation $t_0(yw^{-1}) = v$. Let $yw^{-1} = w_v \cdots w_2w_1$ be a standard decomposition of yw^{-1} into cyclic components with $\pi(w_i) = \eta_i$ for $i \in [v]$. Then the equation $t_0(yw^{-1}) = v$ implies that $w_i \in G(m, m, n)$ for any $i \in [v]$. So Condition (3.10.2) follows by (3.10.3). □

3.11. Algorithms

In the following (1)–(5), let $w = [a_1, \ldots, a_n \mid \sigma]$ be in $G(m, m, n)$.

(1) The proof of Theorem 2.10 in 3.8 actually provides an algorithm to find, for any $w \in G(m, m, n)$ with $t_m(w) = 1$, all the elements in $B_2^{(m)}(w) \cap \pi^{-1}(\tau)$ for any $\tau \in U(\pi(w))$. The algorithm can be extended to the case where w is an arbitrary element in $G(m, m, n)$ (i.e., without the restriction of $t_m(w) = 1$) by Lemma 3.3. Write $\tau\sigma^{-1}$ as in (3.10.1). Note that any $y \in B_2^{(m)}(w) \cap \pi^{-1}(\tau)$ satisfies

$$\mathrm{dist}_m(y, w) = l_{mm}(w) - l_{mm}(y) = l_{11}(\tau) - l_{11}(\sigma) = \mathrm{dist}_1(\tau, \sigma).$$

So by Lemma 3.10, the set $B_2^{(m)}(w) \cap \pi^{-1}(\tau)$ consists of all the elements $y = [c_1, \ldots, c_n \mid \tau]$ satisfying $t_m(y) = t_m(w)$ and Condition (3.10.2).

(2) Let $\tau \in S_n$ satisfy $\tau \preceq_{11} \sigma$ and $u := r(\tau) - r(\sigma) > 0$. Write $\tau\sigma^{-1}$ as in (3.10.1). Note that any $y \in B_1^{(m)}(w) \cap \pi^{-1}(\tau)$ satisfies

$$\mathrm{dist}_m(y, w) = l_{mm}(w) - l_{mm}(y) = l_{11}(\sigma) - l_{11}(\tau) = \mathrm{dist}_1(\tau, \sigma).$$

So by Theorem 2.7 and Lemma 3.10, $B_1^{(m)}(w) \cap \pi^{-1}(\tau)$ consists of all the elements $y = [c_1, \ldots, c_n \mid \tau] \in G(m, m, n)$ satisfying $t_m(y) = t_m(w) + u$ and Condition (3.10.2).

In the following (3)–(5), let $y = [c_1, \ldots, c_n \mid \tau]$ be in $G(m, m, n)$.

(3) Suppose $\sigma \preceq_{11} \tau$. We want to check the relation $y \preceq_2 w$. We need first to check the equation $t_m(y) = t_m(w)$. If $t_m(y) \neq t_m(w)$ then $y \not\preceq_2 w$ by Theorem 2.6 and Lemma 3.3. Now assume $t_m(y) = t_m(w)$. Write $\tau\sigma^{-1}$ as in (3.10.1). Then by Theorem 2.6, Lemma 3.3 and Lemma 3.10, we see that $y \preceq_2 w$ if and only if Condition (3.10.2) holds.

(4) Suppose $\tau \preceq_{11} \sigma$, we want to check the relation $y \preceq_1 w$. We need first to check if the equation

$$t_m(y) = t_m(w) + r(\tau) - r(\sigma) \tag{3.11.1}$$

holds. If not then $y \not\preceq_1 w$ by Theorem 2.7. Now assume (3.11.1) holds. Write $\tau\sigma^{-1}$ as in (3.10.1). Then by Theorem 2.7 and Lemma 3.10, we see that $y \preceq_1 w$ if and only if Condition (3.10.2) holds.

(5) Suppose $l_{mm}(w) - l_{mm}(y) = l_{11}(\tau\sigma^{-1})$, i.e., $t_m(y) - t_m(w) = \frac{1}{2}[n + r(\tau) - r(\sigma) - r(\tau\sigma^{-1})]$ by Theorem 1.7. Then by Lemma 3.10, we see that $y \preceq_{mm} w$ if and only if Condition (3.10.2) holds.

Note that (5) includes (3)–(4) as its special cases.

Remark 3.12. The cardinal of the set $B_1^{(m)}(w) \cap \pi^{-1}(\tau)$ depends on w, τ, not just on $\pi(w), \tau$. For example, let $\sigma = (123)(45)(67)(8)$, $\tau = (2)(13)(45)(67)(8)$ in S_8 and $w_1 = [2, 4, 5, 3, 6, 10, 11, 7 \mid \sigma]$, $w_2 = [2, 3, 5, 4, 6, 10, 11, 7 \mid \sigma]$ in $G(12, 12, 8)$. Then $\tau \lessdot_{11} \sigma$. Any $x \in B_1^{(12)}(w_i) \cap \pi^{-1}(\tau)$, $i = 1, 2$, has the form $x = t(1, 2; a) \cdot w_i$ for some $a \in [12]$. By Theorem 2.7, there are 6 elements of the form $t(1, 2; a) \cdot w_1$ in $B_1^{(12)}(w_1) \cap \pi^{-1}(\tau)$ with $a \in \{2, 3, 6, 8, 9, 11\}$ and 7 elements of the form $t(1, 2; a) \cdot w_2$ in $B_1^{(12)}(w_2) \cap \pi^{-1}(\tau)$ with $a \in \{2, 4, 6, 7, 9, 11, 0\}$.

4. The enumeration of the set $U(\sigma)_k$

In the present section, we shall enumerate the cardinal of the set $U(\sigma)_k$ occurring in Theorem 2.10 for any $\sigma \in S_n$ and any $k \in [r(\sigma)]$ (i.e., Theorem 4.4). This result is of independent combinatorial interest.

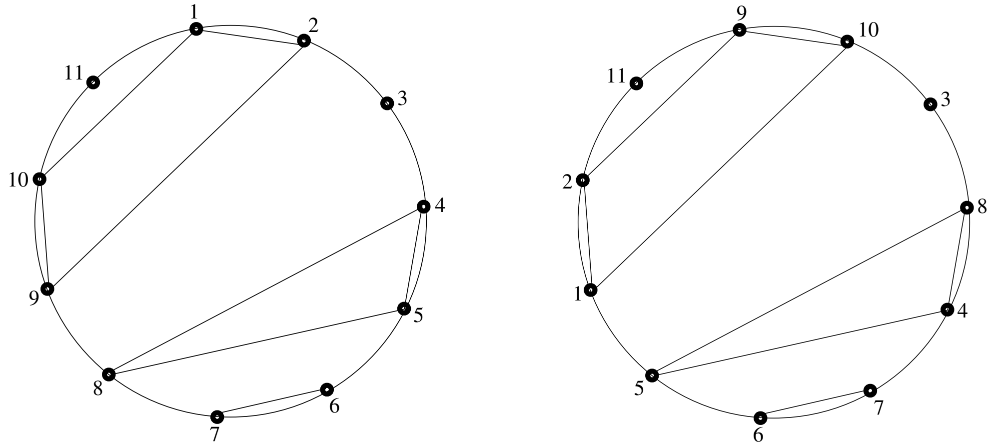
4.1. Some symmetric functions

Let $\mathbb{Z}[\mathbf{x}] = \mathbb{Z}[x_1, x_2, x_3, \ldots]$ be the polynomial ring in variables $x_1, x_2, x_3, \ldots$ with integer coefficients. For any $E \subset \mathbb{P}$ with $t = |E| < \infty$, let $\Pi(\mathbf{x}; E) = \prod_{i \in E} x_i$ and $\Sigma(\mathbf{x}; E) = \sum_{i \in E} x_i$.

For any $t \in \mathbb{P}$, define

$$F(\mathbf{x}; t) = F(x_1, \ldots, x_t) = \begin{cases} 1, & \text{if } t = 1; \\ \Pi(\mathbf{x}; [t]) \left(\prod_{k=1}^{t-2} (\Sigma(\mathbf{x}; [t]) - k) \right), & \text{if } t \geqslant 2. \end{cases} \tag{4.1.1}$$

This is a symmetric polynomial in variables $x_1, x_2, \ldots, x_t$. For $k \in [t]$, let $\Lambda(\mathbf{x}; t, k)$ be the set of all partitions $P = \{P_1, \ldots, P_k\}$ of the set $\{x_1, \ldots, x_t\}$ into k non-empty parts P_i. For any $P = \{P_1, \ldots, P_k\} \in \Lambda(\mathbf{x}; t, k)$, define



(a) (b)

Fig. 1.

$F(\mathbf{x}; P) = \prod_{i=1}^{k} F(P_i)$, where $F(P_i) = F(x_{h_1}, \ldots, x_{h_q})$ if $P_i = \{x_{h_1}, \ldots, x_{h_q}\}$, $F(P_i)$ is well defined since the variables $x_{h_1}, \ldots, x_{h_q}$ are symmetric in it. For any multi-set $\mathbf{m} = [\![m_1, \ldots, m_t]\!]$, choose an arrangement $\overline{\mathbf{m}} = (m_1, m_2, \ldots, m_t)$. Let $F(\overline{\mathbf{m}}; P)$ (respectively, $F(\mathbf{m}; k)$) be obtained from $F(\mathbf{x}; P)$ (respectively, $F(\mathbf{x}; t, k) = \sum_{P \in \Lambda(\mathbf{x}; t, k)} F(\mathbf{x}; P)$) by substitution of $(x_1, \ldots, x_t) = (m_1, \ldots, m_t)$. Note that although $F(\overline{\mathbf{m}}; P)$ depends on the choice of an arrangement $\overline{\mathbf{m}}$, the value $F(\mathbf{m}; k)$ is independent of such a choice for $\mathbf{m}$ and hence is well defined.

Example 4.2. Let $P = \{\{x_1, x_4, x_6\}, \{x_2, x_3, x_5, x_7\}\} \in \Lambda(\mathbf{x}; 7, 2)$. We have

$$\begin{aligned} F(\mathbf{x}; P) &= F(x_1, x_4, x_6) \cdot F(x_2, x_3, x_5, x_7) \\ &= \left(\prod_{i=1}^{7} x_i\right)(x_1 + x_4 + x_6 - 1)(x_2 + x_3 + x_5 + x_7 - 1)(x_2 + x_3 + x_5 + x_7 - 2). \end{aligned}$$

Suppose that $\mathbf{m} = [\![2, 3, 3, 4, 5, 5, 6]\!]$ and $\overline{\mathbf{m}} = (2, 3, 3, 4, 5, 5, 6)$. Then $F(\overline{\mathbf{m}}; P) = 25{,}920{,}000$ and $F(\mathbf{m}; 2) = 1{,}520{,}928 + 10{,}800 \times (125{,}334 + 78{,}690) = 1520{,}928 + 2{,}203{,}459{,}200 = 2{,}204{,}980{,}128$, where $1{,}520{,}928$ (respectively, $10{,}800 \times 125{,}334$; $10{,}800 \times 78{,}690$) is the sum of all the $F(\overline{\mathbf{m}}; P)$'s with P ranging over all the partitions of $\{x_1, \ldots, x_7\}$ of the form $P = \{P_1, P_2\}$ with $|P_1| = 1$ (respectively, $|P_1| = 2$; $|P_1| = 3$).

4.3. The multi-set $m(\sigma)$

In the remaining part of the section, we fix

$$\sigma = (i_{11}, i_{12}, \ldots, i_{1m_1})(i_{21}, i_{22}, \ldots, i_{2m_2}) \cdots (i_{r1}, i_{r2}, \ldots, i_{rm_r}) \in S_n, \tag{4.3.1}$$

a product of disjoint cyclic permutations $\sigma_j = (i_{j1}, i_{j2}, \ldots, i_{jm_j})$ with $\sum_{j=1}^{r} m_j = n$ and $m_j \in \mathbb{P}$. Set $\mathbf{m}(\sigma) = [\![m_1, \ldots, m_r]\!]$.

Theorem 4.4. *In the above setup, we have $|U(\sigma)_k| = F(\mathbf{m}(\sigma); k)$ for any $k \in [r]$.*

In order to prove Theorem 4.4, we need some preparation.

4.5. Noncrossing partitions of a cyclic permutation

For a k-cyclic permutation $\tau = (i_1, i_2, \ldots, i_k) \in S_n$ with $k \in [n]$, define a circle with k nodes labelled by the numbers $i_1, i_2, \ldots, i_k$ respectively in clockwise order, call such a circle a *k-circle*, denoted by $\mathcal{O}(\tau)$. Partition the k nodes of $\mathcal{O}(\tau)$ into h non-empty parts $X_1, \ldots, X_h$ such that the convex hulls $\overline{X}_j$ of the parts X_j, $j \in [h]$, on the k-circle $\mathcal{O}(\tau)$ are pairwise disjoint (combinatorially, this amounts to saying that there do not exist $i_a, i_c \in X_j$ and $i_b, i_d \in X_l$ satisfying $a < b < c < d$ for any $j \neq l$ in $[h]$). The partition $X = \{X_1, \ldots, X_h\}$ is called *a noncrossing partition of $\mathcal{O}(\tau)$*. Let $P(\tau)$ be the set of all noncrossing partitions of $\mathcal{O}(\tau)$. To any $X = \{X_1, \ldots, X_h\} \in P(\tau)$, we associate an element $\phi(X)$ in S_n as follows. Reading the nodes clockwise along the boundary of $\overline{X}_j$, we get a cyclic permutation ϕ_j. Then set $\phi(X) = \phi_1\phi_2 \cdots \phi_h$. This defines a map $\phi : P(\tau) \to S_n$, which is injective.

For example, let $n \geqslant 11$. For $\tau = (1, 2, \ldots, 11) \in S_n$, we take a partition X of $\mathcal{O}(\tau)$ as in Fig. 1(a). Then the corresponding $\phi(X) \in S_n$ is $(1, 2, 9, 10)(4, 5, 8)(6, 7)(3)(11)$.

4.6. σ-dominant n-cycles

Call an n-cyclic permutation $\tau \in S_n$ *σ-dominant*, if $\sigma = \phi(X)$ for some $X \in P(\tau)$. Hence an n-cyclic permutation $\tau = (i_1, i_2, \ldots, i_n) \in S_n$ is σ-dominant if and only if the following two conditions hold:

(i) The nodes $i_{j1}, i_{j2}, \ldots, i_{jm_j}$ ($j \in [r]$) arrange on the n-circle $\mathcal{O}(\tau)$ in clockwise order.
(ii) For any $h \neq k$ in $[r]$, there does not exist any $a < b$ in $[m_h]$ and any $c < d$ in $[m_k]$ such that the chord of $\mathcal{O}(\tau)$ connecting i_{ha}, i_{hb} intersects that connecting i_{kc}, i_{kd}.

When an n-cyclic permutation $\tau \in S_n$ satisfies conditions (i)–(ii), call the part $I_j(\tau) = \{i_{j1}, i_{j2}, \ldots, i_{jm_j}\}$ ($j \in [r]$) a *σ-part* of τ. Note that as a σ-part of an n-circle $\mathcal{O}(\tau)$, $I_j(\tau)$ is a node set on $\mathcal{O}(\tau)$, rather than only a label set, that is, we are also concerned with the relative positions of those nodes on $\mathcal{O}(\tau)$ besides their labels.

Then $U(\sigma)_1$ is exactly the set of all the σ-dominant n-cyclic permutations τ in S_n, here and later we identify any $\tau \in U(\sigma)_1$ with the corresponding n-circle $\mathcal{O}(\tau)$.

Let us read the nodes clockwise around an n-circle $\mathcal{O}(\tau) \in U(\sigma)_1$. Each σ-part I of $\mathcal{O}(\tau)$ can be decomposed into a disjoint union of segments, where by a segment of I, we mean a maximal subset of nodes of $\mathcal{O}(\tau)$ in I which are located on the circle $\mathcal{O}(\tau)$ consecutively. A σ-part I of $\mathcal{O}(\tau)$ is called *connected* if I consists of a single segment.

For example, on the 11-circle $\mathcal{O}(\tau)$ in Fig. 1(a), there are three connected σ-parts: $\{3\}, \{11\}, \{6, 7\}$; there are two segments $\{1, 2\}$ and $\{9, 10\}$ (respectively, $\{4, 5\}$ and $\{8\}$) in the σ-part $I = \{1, 2, 9, 10\}$ (respectively, $I = \{4, 5, 8\}$).

A key observation is that any σ-dominant n-circle $\mathcal{O}(\tau)$ contains at least one connected σ-part. In particular, when $r(\sigma) \leqslant 2$, the σ-parts of any σ-dominant n-circle are all connected.

4.7. σ-isotopical n-circles

Two n-circles $\mathcal{O}(\tau), \mathcal{O}(\tau') \in U(\sigma)_1$ are said *σ-isotopical* if for any $j \in [r]$, the σ-part $I_j(\tau')$ of $\mathcal{O}(\tau')$ can be obtained from the σ-part $I_j(\tau)$ of $\mathcal{O}(\tau)$ by the action of some power of the cyclic factor σ_j of σ. For example, two 11-circles in Fig. 1 are isotopical. Being σ-isotopical is an equivalence relation on $U(\sigma)_1$. When $r = r(\sigma) > 1$, the number of elements in each σ-isotopical class $\overline{E}$ of $U(\sigma)_1$ is $n(\sigma) = \prod_{i=1}^{r} m_i$, this number depends only on σ but not on the choice of $\overline{E}$.

For example, $n(\sigma) = 24$ for $\sigma = (1, 2, 9, 10)(4, 5, 8)(6, 7)(3)(11) \in S_{11}$.

4.8. The map $\phi_E : U(\sigma)_{1,E} \to U(\sigma_E)_1$

Fix $E \subseteq [r]$. Set $E^c = [r] \setminus E$. Let σ_E be the permutation on the set $\cup_{j\in E} I_j$ which is obtained from σ by removing all the cyclic factors σ_h, $h \in E^c$ (see (4.3.1)). Denote by $n_E(\sigma)$ the cardinal for the set $\overline{U(\sigma_E)}_1$ of all isotopical classes of $U(\sigma_E)_1$, denote $m_E(\sigma) = \sum_{j\in E} |I_j|$ and

$$p_E(\sigma) = \prod_{k=0}^{|E^c|-1} (m_E(\sigma) + k), \tag{4.8.1}$$

where we stipulate $p_{[r]}(\sigma) = 1$. Let $U(\sigma)_{1,E}$ be the set of all $\mathcal{O}(\tau) \in U(\sigma)_1$ such that all the σ-parts $I_h(\tau)$, $h \in E^c$, are connected. In particular, $U(\sigma)_{1,[r]} = U(\sigma)_1$. Then there exists a map $\phi_E : U(\sigma)_{1,E} \to U(\sigma_E)_1$ such that for any $\mathcal{O}(\tau) \in U(\sigma)_{1,E}$, the circle $\phi_E(\mathcal{O}(\tau)) = \mathcal{O}(\tau_E) \in U(\sigma_E)_1$ is obtained from $\mathcal{O}(\tau)$ by removing all the (nodes in the) σ-parts $I_h(\tau)$, $h \in E^c$.

It is obvious that ϕ_E is surjective. Observe the following facts on $E \subseteq [r]$:

(1) The set $U(\sigma)_{1,E}$ is a union of some isotopical classes of $U(\sigma)_1$;
(2) The map ϕ_E induces a surjective map $\overline{\phi}_E$ from the set $\overline{U(\sigma)}_{1,E}$ of isotopical classes of $U(\sigma)_1$ in $U(\sigma)_{1,E}$ to the set $\overline{U(\sigma_E)}_1$ of isotopical classes of $U(\sigma_E)_1$.
(3) For $\mathcal{O}(\tau) \in U(\sigma)_1$, let E_τ be the set of all $i \in [r]$ such that the σ-parts $I_i(\tau)$ of $\mathcal{O}(\tau)$ are connected. Then $\tau \in U(\sigma)_{1,E}$ if and only if $E_\tau \supseteq E^c$.

Now we consider the set $\phi_E^{-1}(\mathcal{O}(\tau_E))$ for any $E \subseteq [r]$ and any $\mathcal{O}(\tau_E) \in U(\sigma_E)_1$. Clearly, $\phi_{[r]}^{-1}(\mathcal{O}(\tau_{[r]})) = \{\mathcal{O}(\tau_{[r]})\}$ is a singleton. Now assume $E \subset [r]$. Let $E^c = \{i_1, \ldots, i_t\}$ be with $t = |E^c|$. First we insert the σ-part I_{i_1} on the circle $\mathcal{O}(\tau_E)$ in one of the $m_E(\sigma)$ different ways (Note that $\mathcal{O}(\tau_E)$ contains $m_E(\sigma)$ nodes) such that the resulting circle $\mathcal{O}(\tau_E^{(1)})$ has its σ-part I_{i_1} connected. Secondly, we insert the σ-part I_{i_2} on the circle $\mathcal{O}(\tau_E^{(1)})$ in one of the $m_E(\sigma) + 1$ different ways such that the resulting circle $\mathcal{O}(\tau_E^{(2)})$ has its σ-parts I_{i_1}, I_{i_2} connected, etc. Finally, we insert the σ-part I_{i_t} on the circle $\mathcal{O}(\tau_E^{(t-1)})$ in one of the $m_E(\sigma) + t - 1$ different ways such that the resulting circle $\mathcal{O}(\tau_E^{(t)})$ has its σ-parts $I_{i_1}, I_{i_2}, \ldots, I_{i_t}$ connected. Hence $\mathcal{O}(\tau_E^{(t)}) \in U(\sigma)_{1,E}$ and $\phi_E(\mathcal{O}(\tau_E^{(t)})) = \mathcal{O}(\tau_E)$. This implies that for any $E \subseteq [r]$ and any $\overline{\mathcal{O}(\tau_E)} \in \overline{U(\sigma_E)}_1$, the inverse image $\overline{\phi}_E^{-1}(\overline{\mathcal{O}(\tau_E)})$ has cardinal $p_E(\sigma)$ (see (4.8.1)). So $n_E(\sigma)p_E(\sigma) = |\overline{U(\sigma_E)}_{1,E}|$. Therefore by the inclusion–exclusion principle, we get

$$|\overline{U(\sigma)}_1| = \sum_{E\subset[r]} (-1)^{|E^c|-1} n_E(\sigma)p_E(\sigma). \tag{4.8.2}$$

4.9. Proof of Theorem 4.4

By Theorem 1.7, we see that $\tau \in S_n$ is in $U(\sigma)_k$ if and only if τ can be expressed as a product $\tau = \tau_1\tau_2\cdots\tau_k$ of disjoint cyclic permutations such that there exists some partition $P = \{P_1, \ldots, P_k\}$ of $[r]$ into k non-empty parts with $\tau_j \in U(\overline{\sigma}_j)_1$, where $\overline{\sigma}_j = \prod_{h\in P_j} \sigma_h$ for any $j \in [k]$. So we need only to consider the case $k = 1$ and to prove Eq. (4.9.1) below:

$$|U(\sigma)_1| = \begin{cases} \prod_{j=1}^{r} m_j \prod_{k=1}^{r-2}(n-k), & \text{if } r \geqslant 2; \\ 1, & \text{if } r = 1. \end{cases} \tag{4.9.1}$$

The result is obvious for $r = 1$. Now assume $r \geqslant 2$. Since each isotopical class of $U(\sigma)_1$ contains $\prod_{j=1}^{r} m_j$ elements, to prove (4.9.1), it is enough to prove the following

$$|\overline{U(\sigma)}_1| = \prod_{k=1}^{r-2}(n-k). \tag{4.9.2}$$

Apply induction on $r \geqslant 2$. Eq. (4.9.2) is obvious for $r = 2$. Now assume $r > 2$. By inductive hypothesis, we have $n_E(\sigma) = \prod_{k=1}^{|E|-2}(m_E(\sigma) - k)$ for any $E \subset [r]$ with $|E| \geqslant 2$. So $n_E(\sigma)p_E(\sigma) = \prod_{k=1}^{r-2}(m_E(\sigma) + |E^c| - k)$ by (4.8.1). Thus by (4.8.2), to prove (4.9.2), we need only to show the following

$$\prod_{k=1}^{r-2}(n-k) = \sum_{E\subset[r]} (-1)^{|E^c|-1} \prod_{k=1}^{r-2}(m_E(\sigma) + |E^c| - k), \tag{4.9.3}$$

or more generally, to show the following equation of symmetric polynomials in $x_1, \ldots, x_r$ (see 4.1):

$$\prod_{k=1}^{r-2}(\Sigma(\mathbf{x}; [r]) - k) = \sum_{E\subset[r]} (-1)^{|E^c|-1} \prod_{k=1}^{r-2}(\Sigma(\mathbf{x}; E) + |E^c| - k). \tag{4.9.4}$$

Note that (4.9.3) is a special case of (4.9.4) by setting $(x_1, \ldots, x_r) = (m_1, \ldots, m_r)$. By replacing $x_1, \ldots, x_r$ by $x_1+1, \ldots, x_r+1$ respectively, (4.9.4) becomes

$$\sum_{E\subseteq[r]} (-1)^{|E^c|} \prod_{k=2}^{r-1}(\Sigma(\mathbf{x}; E) + k) = 0. \tag{4.9.5}$$

Let us consider the following equation of symmetric polynomials in $x_1, \ldots, x_r, y_1, \ldots, y_{r-2}$:

$$\sum_{E\subseteq[r]} (-1)^{|E^c|} \prod_{k=1}^{r-2}(\Sigma(\mathbf{x}; E) + y_k) = 0. \tag{4.9.6}$$

Note that (4.9.5) is a special case of (4.9.6) with $(y_1, y_2, \ldots, y_{r-2}) = (2, 3, \ldots, r-1)$.

Now we want to prove (4.9.6). Regarding it as an equation of the symmetric polynomial in $y_1, \ldots, y_{r-2}$, one need only to show that the coefficients of the monomials $y_1y_2\cdots y_k$ for $k \in [0, r-2]$ on the left-hand side of (4.9.6) are all zero, where we stipulate $y_1y_2\cdots y_k = 1$ for $k = 0$. That is, we show the following equations of symmetric polynomials in $x_1, \ldots, x_r$:

$$\sum_{E\subseteq[r]} (-1)^{|E^c|} \Sigma(\mathbf{x}; E)^t = 0 \quad \text{for any } 0 \leqslant t \leqslant r-2. \tag{4.9.7}$$

We need only to show that the coefficients $c_{\mathbf{k}}$ of the monomials $x_1^{k_1}x_2^{k_2}\cdots x_u^{k_u}$ on the left-hand side of (4.9.7) are all zero, where $\mathbf{k} = (k_1, \ldots, k_u)$ ranges over all the u-tuples over $\mathbb{P}$ with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_u > 0$ and $k_1 + \cdots + k_u = t$. Set

$$\binom{t}{k_1, \ldots, k_u} = \frac{t!}{k_1!k_2!\cdots k_u!(t-k_1-k_2-\cdots-k_u)!}.$$

Then we have

$$\begin{aligned} c_{\mathbf{k}} &= \binom{t}{k_1, \ldots, k_u} \sum_{[u]\subseteq E\subseteq[r]} (-1)^{|E^c|} \\ &= (-1)^{r+u}\binom{t}{k_1, \ldots, k_u} \sum_{l\in[u,r]} (-1)^{l-u}\binom{r-u}{l-u} \\ &= (-1)^{r+u}\binom{t}{k_1, \ldots, k_u}(1-1)^{r-u} = 0 \end{aligned} \tag{4.9.8}$$

by the fact of $u < r$.

This implies Eq. (4.9.6). So Theorem 4.4 is proved. □

Example 4.10. Let $\sigma = (1, 2, 3)(4, 5, 6)(7, 8, 9, 10) \in S_{10}$. Then $\mathbf{m}(\sigma) = [\![3, 3, 4]\!]$. By Theorem 4.4, we get

$$|U(\sigma)_2| = F(\mathbf{m}(\sigma); 2) = F(3, 3)F(4) + F(3, 4)F(3) + F(3, 4)F(3) = 33$$

and

$$|U(\sigma)_1| = F(\mathbf{m}(\sigma); 1) = F(3, 3, 4) = 324.$$

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